Modeling Propositional Logic with FOL

Intended Interpretations

True(x,y) - x is true on the truth-table row y
False(x,y) - x is false on the truth-table row y
Neg(x,y) - x is the negation of y
Conj(x,y,z) - x is the conjunction of y and z
Disj(x,y,z) - x is the disjunction of y and z
Arrow(x,y,z) - x is a conditional with y as the antecedent and z as the consequent
Darrow(x,y,z) - x is a biconditional with y as the left side and z as the right side
Taut(x) - x is a tautology
Contra(x) - x is a contradiction
Contin(x) - x is contingent
ProveFrom(x,y) - x is provable from y
Entails(x,y) - x entails y
NoPremises(x) - x is provable from no premises

Some Axioms

Here are twelve true facts about propositional logic

1. **$\forall x \forall y(\text{True}(x,y) \leftrightarrow \neg\text{False}(x,y))$**
   Formulas are either true or false on a row of a truth-table and never both

2. **$\forall x \forall y(\text{Neg}(x,y) \leftrightarrow \forall z(\text{True}(x,z) \leftrightarrow \text{False}(x,z))$**
   Truth table definition of negation

3. **$\forall x \forall y \forall z[\text{Conj}(x,y,z) \leftrightarrow \forall w(\text{True}(x,w) \leftrightarrow (\text{True}(y,w) \wedge \text{True}(z,w)))]$**
   Truth table definition of conjunction

4. **$\forall x \forall y \forall z[\text{Disj}(x,y,z) \leftrightarrow \forall w(\text{True}(x,w) \leftrightarrow (\text{True}(y,w) \vee \text{True}(z,w)))]$**
   Truth table definition of disjunction

5. **$\forall x \forall y \forall z[\text{Arrow}(x,y,z) \leftrightarrow \forall w(\text{True}(x,w) \leftrightarrow (\text{True}(y,w) \rightarrow \text{True}(z,w)))]$**
   Truth table definition of conditional

6. **$\forall x \forall y \forall z[\text{Darrow}(x,y,z) \leftrightarrow \forall w(\text{True}(x,w) \leftrightarrow (\text{True}(y,w) \leftrightarrow \text{True}(z,w)))]$**
   Truth table definition of biconditional

7. **$\forall x(\text{Taut}(x) \leftrightarrow \forall y\ \text{True}(x,y))$**
   Tautologies are true on every row of a truth-table

8. **$\forall x(\text{Contra}(x) \leftrightarrow \forall y\ \text{False}(x,y))$**
   Contradictions are false on every row of a truth-table

9. $\forall$**x[Contin(x)** $\leftrightarrow$ **(**$\exists$**y False(x,y)** $\wedge$ $\exists$**y True (x,y))]**
Contingent sentences are true on some rows and false on others

10. $\forall$**x**$\forall$**y(Entails(x,y)** $\leftrightarrow$ **ProveFrom(y,x))**
One formula entails another if and only if the second is provable from the first (soundness and completeness theorems)

11. $\forall$**x**$\forall$**y[Entails(x,y)** $\leftrightarrow$ $\forall$**z(True(x,z)** $\rightarrow$ **True(y,z)))]**
One formula entails another if and only if every TVA that makes the first formula is true also makes the second formula true

12. $\forall$**x(NoPremises(x)** $\leftrightarrow$ $\forall$**y ProveFrom(x,y))**
A formula is provable from no premises if and only if it is provable from any premise

Using these twelve axioms we can prove many true things about propositional logic such as:

- The paradoxes of material implication are provable from no premises
- Tautologies are provable from no premises
- Given two formulas, if neither is provable from the other, both are contingent
- Contradictions allow you to prove any formula
- Tautologies are provable from any formula
- DeMorgan's Law is valid
- Modus Tollens is valid
- Provability is transitive
- If a formula is provable from its own negation, then its negation is a contradiction
- If a biconditional is provable from no premises, then if the left side is a contradiction, then so is the right side

While many things are provable, some things aren't. Here is something that isn't:

If a formula is provable from any formula at all, then it is a tautology
$\forall$**x(**$\forall$**y ProveFrom(x,y)** $\rightarrow$ **Taut(x))**

Question: Since that formula is not provable from those premises, then there is some true fact about propositional logic that is not represented in those premises. What fact is it that is left out?